Discrete Mathematics: Combinatorics and Graph Theory

Homework 3: Due 12/2

Instructions. Solve any 10 questions. Typeset or write neatly and show your work to receive full credit.

1. Find a formula for a_n given the stated recurrence relation and initial values:
   (a) $a_n = 3a_{n-1} - 2$ for $n \geq 1$ with $a_0 = 1$.
   (b) $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 2$ with $a_0 = 1$, $a_1 = 8$.
   (c) $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 2$ with $a_0 = a_1 = 1$.
   (d) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$ with $a_0 = 1$, $a_1 = 3$.
   (e) $a_n = 3a_{n-1} - 1$ for $n \geq 1$ with $a_0 = 1$.

2. Find a recurrence relation for the number of ternary strings of length n that contain either two consecutive 0s or two consecutive 1s.
   (a) What are the initial conditions?
   (b) How many ternary strings of length six contain two consecutive 0s or two consecutive 1s?

3. A bus driver pays all tolls, using only nickels and dimes, by throwing one coin at a time into the mechanical toll collector.
   (a) Find a recurrence relation for the number of different ways the bus driver can pay a toll of n cents (where the order in which the coins are used matters).
   (b) In how many different ways can the driver pay a toll of 45 cents?

4. Show that the Fibonacci numbers satisfy the recurrence relation $f_n = 5f_{n-4} + 3f_{n-5}$ for $n = 5, 6, 7, \cdots$, together with the initial conditions $f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2$, and $f_4 = 3$. Use this recurrence relation to show that f_{5n} is divisible by 5, for $n = 1, 2, 3, \cdots$.

5. Solve the recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ with $a_0 = -5, a_1 = 4$, and $a_2 = 88$.

6. What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has the roots $-1, -1, -1, 2, 2, 5, 5, 7$?

7. Find the solution of the recurrence relation $a_n = 2a_{n-1} + 3 \cdot 2^n$.

8. Find the solution of the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2} + 2^n + n + 3$ with $a_0 = 1$ and $a_1 = 4$.

9. Suppose $\langle a \rangle$ satisfies the recurrence $a_n = -a_{n-1} + \lambda^n$. Determine the values of λ such that $\langle a \rangle$ can be unbounded.

10. Let $a_n = n^3$. Find a constant-coefficient first-order linear recurrence relation satisfied by $\langle a \rangle$. Does there exist a homogeneous constant-coefficint first-order linear recurrence relation satisfied by $\langle a \rangle$? Why or why not?

11. Derive a general formula for the recurrence $a_n = ca_{n-1} + f(n)\beta^n$ where f is a polynomial and β a constant.

12. Let f be a polynomial of degree n. The *first difference* of f is the function $g = \Delta f$ defined by $g(x) = f(x+1) - f(x)$. The *k-th difference* of f is the function $g^{(k)}$ defined inductively by $g^{(0)} = f$ and $g^{(k)} = \Delta g^{(k+1)}$ for $k \geq 1$. Obtain a formula for the nth difference of f.